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SELF-INJECTIVE AND PF ENDOMORPHISM RINGS

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ABSTRACT
We study the endomorphism ring S of a Σ-quasiprojective module M, giving necessary and sufficient conditions on M for S to have certain properties, such as, e.g., being QF or left (F)PF.

The endomorphism ring S of a Σ-quasiprojective module M has been studied in [5] by means of a category equivalence between a suitable quotient category of the category $\sigma[M]$ of all R-modules subgenerated by M and the quotient category of S-Mod by the left Gabriel topology $\mathcal{F}$ of S given by $\mathcal{F} = \{I \subset {}_S S \mid MI = M\}$. This study was based on the idea that this quotient category $(S, \mathcal{F})$-Mod carries a lot of information about S (or S-Mod), mainly due to the facts that S is an object of $(S, \mathcal{F})$-Mod and the inclusion functor of $(S, \mathcal{F})$-Mod in S-Mod is exact, which, in particular, imply that finitely generated $\mathcal{F}$-torsionfree left S-modules (and among them, finitely generated left ideals) belong to $(S, \mathcal{F})$-Mod. Then we may transfer properties of S-Mod through the equivalence and interpret them in terms of M or of R-modules closely related to M. In the present paper, we continue this investigation and we observe that $(S, \mathcal{F})$-Mod has no less than three realizations up to equivalence as a full subcategory of $\sigma[M]$; not only $(S, \mathcal{F})$-Mod is equivalent to the category $\mathcal{C}[M]$ introduced in [5] ($\mathcal{C}[M]$ is the quotient category of $\sigma[M]$ with respect to the natural torsion theory defined by M) but it is also equivalent to the category GF$[M]$ of M-generated M-faithful modules, and to the category CD$[M]$ of modules of M-codominant dimension $\geqq 2$. Since these categories have different properties in which regard their relationship with $\sigma[M]$ and R-Mod, we use whichever is more convenient in order to study a specific property.

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The construction of $\mathscr{C}[M]$ for a Σ-quasiprojective module M suggests the possibility of extending it to an arbitrary module. This is accomplished by considering the (hereditary) torsion theory of $\sigma[M]$ generated by all the modules of the form X/X_M, where X is a module of $\sigma[M]$ and X_M is the largest M-generated submodule of X. Then we may consider the quotient category of $\sigma[M]$ by this torsion theory, which we still call $\mathscr{C}[M]$. $\mathscr{C}[M]$ is a Grothendieck category and the image $\mathrm{T}(M)$ of M by the canonical functor $\mathrm{T}: \sigma[M] \to \mathscr{C}[M]$ is a generator of $\mathscr{C}[M]$. In general, the endomorphism ring of $\mathrm{T}(M)$ is different from the endomorphism ring S of M, but if M satisfies certain additional conditions, such as, e.g., being M-faithful (i.e., torsionfree in $\sigma[M]$) and quasiinjective, then M is isomorphic to $\mathrm{T}(M)$ and we may use the Gabriel–Popescu Theorem to represent $\mathscr{C}[M]$ as a quotient category of S-Mod and, consequently, to study properties of S in terms of M.

In the first section we study M-faithful modules, and we prove that if M is a Σ-quasiprojective module, then the categories GF$[M]$ and CD$[M]$ are equivalent to $(S, \mathscr{F})$-Mod (Theorem 1.3). The second section is devoted to the study of self-injective and quasi-Frobenius (QF) endomorphism rings. By using the category $\mathscr{C}[M]$ we show that if M is M-faithful, then $S = \mathrm{End}(_RM)$ is left self-injective if and only if M is a quasiinjective module (Theorem 2.7). This leads to a characterization of the M-faithful modules M such that S is a quasi-Frobenius ring.

In the last part of the paper we consider several classes of rings which generalize QF rings, such as (left) Kasch rings, PF rings, FPF rings and QF-3′ rings. For instance, we show in Theorem 3.5 that if M is a M-faithful quasiprojective module, then S is left PF if and only if M is a quasiinjective module which is either finitely cogenerated or a finitely generated RZ-module (M is a RZ-module when it cogenerates each of its simple quotients). This extends results of Onodera [14] and Rutter [17]. We also show that, with M as in Theorem 3.5, S is left PF if and only if M is finitely generated and every M-generated module which cogenerates M generates M. For projective modules this condition is quite close to (and weaker than) that of being a PF module in the sense of Page [15]. In fact, we show in Proposition 3.7 that PF modules in the sense of Page are the same as PF modules in the sense of Rutter [17].

FPF endomorphism rings (R is left FPF if each finitely generated faithful R-module is a generator of R-Mod) have been recently studied in [15], where a sufficient condition is given for the endomorphism ring of a finitely generated projective module P to be left FPF, showing also that if P is a self-generator, then that condition is also necessary [15, Theorem 4]. We extend this result to

Σ-quasiprojective modules and show that the converse holds without the assumption of M being a finitely generated self-generator (Theorem 3.10).

Finally, we study when the endomorphism ring of a Σ-quasiprojective module is a left QF-3′ ring, generalizing results of [13].

Throughout this paper R denotes an associative ring with identity and R-Mod the category of left R-modules. All modules will be left R-modules, unless otherwise stated. If M is a module, then we will say that a module N is (finitely) M-generated if it is a quotient of a (finite) direct sum $M^{(I)}$ of copies of M. When every submodule of M is M-generated, M is called a self-generator. The full subcategory of R-Mod consisting of the submodules of M-generated modules (M-subgenerated modules) will be denoted by $\sigma[M]$; it is a Grothendieck category [21]. We will denote by X_M the largest M-generated submodule of a module X.

We recall that a module N is M-projective (M-injective) if, for every quotient module (submodule) X of M, the canonical homomorphism

$$\mathrm{Hom}_R(N, M) \to \mathrm{Hom}_R(N, X) \qquad (\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(X, N))$$

is an epimorphism and, in particular, M is quasiprojective (quasiinjective) when it is M-projective (M-injective). M is a projective object of $\sigma[M]$ if and only if M is Σ-quasiprojective, that is, $M^{(I)}$ is quasiprojective for each set I (see [5]).

If N is a left R-module, then its (Jacobson) radical will be denoted by $\mathrm{Rad}(N)$ and a similar notation will be used for the radical of an object in a Grothendieck category.

In what follows we will denote by $S = \mathrm{End}({}_RM)$ the endomorphism ring of a left R-module M (when dealing with endomorphism rings, we use the convention of writing endomorphisms opposite scalars). There are canonical homomorphisms

$$M \underset{S}{\otimes} M^* \to R \quad \text{and} \quad M^* \underset{R}{\otimes} M \to S \quad (M^* = \mathrm{Hom}_R(M, R))$$

whose images T and T' are two-sided ideals of R and S, respectively. T is the largest M-generated left ideal of R and is called the trace of M. M is called trace-accessible if $TM = M$ [22].

If $\mathscr{F}$ is a left Gabriel topology on a ring R, then the quotient category $(R, \mathscr{F})$-Mod associated with $\mathscr{F}$ is the full subcategory of R-Mod whose objects are the $\mathscr{F}$-closed ($\mathscr{F}$-torsionfree and $\mathscr{F}$-injective) modules. The functor $\mathbf{a}$: R-Mod $\to (R, \mathscr{F})$-Mod which assigns to each module its module of quotients (regarded as an object of $(R, \mathcal{F})$-Mod) is an exact left adjoint of the inclusion functor and hence $(R, \mathcal{F})$-Mod is a Grothendieck category ([19]).

We refer the reader to [2], [7], [16], and [19] for all the ring-theoretic and torsion-theoretic notions used in the text.

1. M-faithful modules

Let M be a left R-module and $\mathcal{T}$ the smallest torsion class of $\sigma[M]$ which contains all the modules of the form X/X_M with X in $\sigma[M]$. Since any submodule of X/X_M is also of this form, it is easy to see in an analogous way to [19, Prop. 3.3, p. 141] that $\mathcal{T}$ is actually a hereditary torsion class of $\sigma[M]$. The $\mathcal{T}$-torsionfree modules of $\sigma[M]$ will be called M-faithful modules and they may be characterized in the following way:

PROPOSITION 1.1. *Let M be a left R-module and N a module of $\sigma[M]$. Then N is M-faithful if and only if, for every nonzero homomorphism $g: X \to N$, with X in $\sigma[M]$, there exists $f: M \to X$ such that $g \circ f \neq 0$.*

PROOF. If N is a M-subgenerated module which is not M-faithful, then there exists X in $\sigma[M]$ such that $\mathrm{Hom}_R(X/X_M, N) \neq 0$. If $h: X/X_M \to N$ is a nonzero homomorphism, composing with the canonical projection from X to X/X_M we get a nonzero homomorphism $g: X \to N$ such that $X_M \subset \mathrm{Ker}\, g$. Therefore, for every $f: M \to X$ we have that $\mathrm{Im}\, f \subset X_M \subset \mathrm{Ker}\, g$ and hence $g \circ f = 0$.

Conversely, assume that there exists a M-subgenerated module X and a nonzero homomorphism $g: X \to N$ such that for every $f: M \to X$, $g \circ f = 0$. Then it is clear that $X_M \subset \mathrm{Ker}\, g$ and this gives a nonzero homomorphism from X/X_M to N, so that N is not M-faithful.

REMARKS. Note that if $\sigma[M] = R$-Mod, then a left R-module N is M-faithful if and only if N is M-distinguished in the sense of [12]. Also, M is a generator of $\sigma[M]$ (i.e., a Σ-self-generator in the terminology of [22]) if and only if every M-subgenerated module is M-faithful. On the other hand, if $R = \mathbf{Z}$ and $M = \mathbf{Q}$, then $\sigma[\mathbf{Q}] = \mathbf{Z}$-Mod and there are no nonzero $\mathbf{Q}$-faithful $\mathbf{Z}$-modules.

In case M is quasiprojective, we have a better description of M-faithful modules:

PROPOSITION 1.2. *Let M be a quasiprojective R-module. Then a M-subgen-*

erated module N is M-faithful if and only if $\mathrm{Hom}_R(M, X) \neq 0$, *for every submodule X of N.*

PROOF. Assume first that N is not M-faithful. Then there exist a M-subgenerated module X and a nonzero homomorphism $f: X/X_M \to N$. Let $p: X \to X/X_M$ be the canonical projection. Since $f \circ p \neq 0$, there exists a finitely generated submodule Z of X such that $(f \circ p)(Z) \neq 0$. Thus we get a homomorphism $q: Z/Z_M \to X/X_M$ such that $f \circ q \neq 0$ and so $Y' = \mathrm{Im}(f \circ q)$ is a nonzero submodule of N which is a quotient of Z. Let $g: Z \to Y'$ be an epimorphism. Since Z is finitely generated, it is isomorphic to a submodule of a finitely M-generated module and thus it is clear that M is Z-projective (using, for instance [2, 16.12]). Therefore, if $h \in \mathrm{Hom}_R(M, Y')$, then h factors through g, $h = g \circ u$ and since $\mathrm{Im}\, u \subset Z_M \subset \mathrm{Ker}\, g$, we see that $h = 0$, so that $\mathrm{Hom}_R(M, Y') = 0$.

Conversely, if there exists a nonzero submodule X of N such that $X_M = 0$, then $\mathrm{Hom}_R(X/X_M, N) = \mathrm{Hom}_R(X, N) \neq 0$ and so N is not M-faithful.

REMARKS. In [5] we called a module N M-faithful if it satisfies that $X_M \neq 0$ for every nonzero submodule X of N. Thus we see that if M is quasiprojective and N is M-subgenerated, both definitions agree.

If M is a Σ-quasiprojective module, then it is easy to see that the class of modules N of $\sigma[M]$ such that $\mathrm{Hom}_R(M, N) = 0$ is a hereditary torsion class of $\sigma[M]$ (see [5]). The corresponding torsionfree class consists of the M-faithful modules of $\sigma[M]$ and thus it follows from Proposition 1.2 that in this case $\mathcal{T} = \{N \in \sigma[M] \mid \mathrm{Hom}_R(M, N) = 0\}$. In the particular case that M is a projective R-module, then we see that the M-faithful modules are precisely the M-distinguished ([12]) modules of $\sigma[M]$. If T denotes the trace of the projective module M on R, then these are just the T-faithful modules in the sense of [18].

If M is a left R-module, we will denote by t the left exact radical of $\sigma[M]$ defined by the hereditary torsion class $\mathcal{T}$ and for each N of $\sigma[M]$ we set $\bar{N} = N/\mathrm{t}(N)$. Thus the M-faithful modules are precisely the M-subgenerated modules N such that $N = \bar{N}$. We will denote by GF[M] the full subcategory of R-Mod determined by the M-generated M-faithful modules. We recall that a module X is said to have M-codominant dimension $\geqq n$, denoted M-cod. dim $X \geqq n$, if there exists an exact sequence $X_n \to \cdots \to X_1 \to X \to 0$, where each X_i is isomorphic to a direct sum of copies of M. CD[M] will be the full subcategory of R-Mod whose objects are the modules of M-codominant dimension $\geqq 2$.

THEOREM 1.3. *Let M be a Σ-quasiprojective module, $S = \mathrm{End}({}_R M)$ and J*

the two-sided ideal of S consisting of the endomorphisms which factor through a finitely generated submodule of M. Then J is an idempotent ideal of S and hence $\mathcal{F} = \{I \subset {}_S S \mid J \subset I\} = \{I \subset {}_S S \mid MI = M\}$ is a left Gabriel topology of S. Moreover, the following assertions hold:

(i) $\mathrm{Hom}_R(M, -): \mathrm{CD}[M] \to (S, \mathcal{F})$-Mod *and* $M \otimes_S -: (S, \mathcal{F})$-Mod $\to$ $\mathrm{CD}[M]$ *are inverse equivalences of categories.*

(ii) $\mathrm{Hom}_R(M, -): \mathrm{GF}[M] \to (S, \mathcal{F})$-Mod *is an equivalence of categories with inverse given by* $Y \to \overline{M \otimes_S Y}$.

In particular, $\mathcal{F} = \{S\}$ *if and only if M is finitely generated and in this case* $\mathrm{Hom}_R(M, -)$ *induces equivalences from* $\mathrm{CD}[M]$ *and* $\mathrm{GF}[M]$ *to* S-Mod.

PROOF. It is clear that J is a two-sided ideal of S. To show that it is idempotent, notice that M, being Σ-quasiprojective, is a direct summand of a direct sum of finitely generated modules (we may take the family of all the finitely generated submodules of M and write M as a quotient of their direct sum). Then, as in the proof of [8, Theorem 2.1], we may find for each $x \in M$ an element $s \in J$ such that $s(x) = x$. This shows that $MJ = M$ and hence that $MJ^2 = M$. Therefore, it is enough to show that J is the smallest left ideal of S with the property that $MJ = M$. Assume that $MI = M$ for $I \subset_S S$. To prove that $J \subset I$, it will suffice to show that if L is a finitely generated submodule of M, then the left ideal $\mathrm{Hom}_R(M, L)$ of S is contained in I. Let $\{x_1, \dots, x_n\}$ be a generating set of L. Since $MI = M$, we may find an $r \geqq 0$ and a homomorphism $f: M^r \to M$ such that $x_i \in \mathrm{Im}\, f$, $i = 1, \dots, n$, and if we denote by $q_j: M \to M^r$ the canonical injections ($j = 1, \dots, r$), then each $f \circ q_j$ belongs to I. Thus there exists a submodule $N \subset M^r$, with canonical injection $u: N \to M^r$ such that $f(N) = L$. If $g \in S$ factors through L, then by the quasiprojectivity of M we get $h: M \to N$ such that $f \circ u \circ h = g$. Calling $p_j: M^r \to M$ to the canonical projections, this means that in S we have $g = \Sigma_1^r (p_j \circ u \circ h)(f \circ q_j)$. Since each $f \circ q_j \in I$, we see that g belongs to I also, and hence $J \subset I$. By [19, Prop. 6.11, p. 150], the set $\mathcal{F} = \{I \subset {}_S S \mid J \subset I\} = \{I \subset {}_S S \mid MI = M\}$ is a left Gabriel topology of S.

To prove (i), consider the functor $\mathrm{Hom}_R(M, -): \mathrm{CD}[M] \to S$-Mod. We claim that it factors through the inclusion functor from $(S, \mathcal{F})$-Mod to S-Mod and hence defines a functor $\mathrm{Hom}_R(M, -): \mathrm{CD}[M] \to (S, \mathcal{F})$-Mod. A left S-module Y is $\mathcal{F}$-closed (i.e., an object of the quotient category $(S, \mathcal{F})$-Mod) if and only if the canonical homomorphism $Y \to \mathrm{Hom}_S(J, Y)$ is an isomorphism [19, Example 3, p. 200]. In order to see that this holds for $Y = \mathrm{Hom}_R(M, N)$, with N a module of $\mathrm{CD}[M]$ (in fact, N could be taken to be any left R-module), we first show that for any left ideal I of S, the canonical homomorphism

$M \otimes_S I \to M$ has torsion kernel in $\sigma[M]$. Assume first that I is finitely generated; then it follows from [8, Lemma 1.1] that the canonical homomorphism $\alpha : I \to \mathrm{Hom}_R(M, M \otimes_S I)$ is an isomorphism. This shows that if $K = \mathrm{Ker}(M \otimes_S I \to M)$, then $\mathrm{Hom}_R(M, K) = 0$ and so K is a torsion module of $\sigma[M]$. Now, if I is an arbitrary left ideal of S, we may write I as a direct limit $I = \varinjlim I_j$ of finitely generated left ideals I_j and if $K_j = \mathrm{Ker}(M \otimes_S I_j \to M)$, we have that the kernel of $M \otimes_S I \to M$ is isomorphic to $\varinjlim K_j$ and hence it is a torsion module of $\sigma[M]$. Therefore, for any M-faithful module X of $\sigma[M]$, there is an isomorphism $\mathrm{Hom}_R(M \otimes_S J, X) \simeq \mathrm{Hom}_R(M, X)$. Thus we have isomorphisms:

$$\mathrm{Hom}_S(J, \mathrm{Hom}_R(M, N)) \simeq \mathrm{Hom}_S(J, \mathrm{Hom}_R(M, \bar{N})) \simeq \mathrm{Hom}_R\left(M \underset{S}{\otimes} J, \bar{N}\right)$$
$$\simeq \mathrm{Hom}_R(M, \bar{N}) \simeq \mathrm{Hom}_R(M, N)$$

which show that $\mathrm{Hom}_R(M, N)$ is a $\mathcal{F}$-closed module.

Next, we are going to show that $\mathrm{Hom}_R(M, -) : \mathrm{CD}[M] \to (S, \mathcal{F})\text{-Mod}$ and $M \otimes_S - : (S, \mathcal{F})\text{-Mod} \to \mathrm{CD}[M]$ are inverse equivalences. If we start with a module N of $\mathrm{CD}[M]$, then we know from [8, Theorem 2.1] that the canonical homomorphism $\beta = \beta_N : M \otimes_S \mathrm{Hom}_R(M, N) \to N$ has torsion kernel and cokernel. Since N is M-generated, β is obviously an epimorphism in this case. On the other hand, using a Schanuel's lemma argument in $\sigma[M]$ to compare the exact sequence $0 \to \mathrm{Ker}\,\beta \to M \otimes_S \mathrm{Hom}_R(M, N) \to N \to 0$ with an exact sequence of the form $0 \to L \to M^{(I)} \to N \to 0$, where L is M-generated, we see that $\mathrm{Ker}\,\beta$ is also M-generated, so that $\mathrm{Ker}\,\beta = 0$ and β is an isomorphism. Now, let Y be a $\mathcal{F}$-closed S-module and $\alpha = \alpha_Y : Y \to \mathrm{Hom}_R(M, M \otimes_S Y)$ the canonical homomorphism. In order to complete the proof of (i), we show that α is an isomorphism. If we call α_* to the homomorphism $M \otimes_S Y \to M \otimes_S \mathrm{Hom}_R(M, M \otimes_S Y)$ obtained by tensoring with M_S, we have that $\beta_{M \otimes Y} \circ \alpha_* = 1$ and since, as we have already seen, $\beta_{M \otimes Y}$ is an isomorphism (for $M \otimes_S Y$ belongs to $\mathrm{CD}[M]$), we have that α_* is also an isomorphism. Then, if $K = \mathrm{Ker}\,\alpha$, the canonical homomorphism $M \otimes_S K \to M \otimes_S Y$ is the zero homomorphism. We have already seen that $M \otimes_S I \to M$ has torsion kernel for every left ideal I of S. Now, by using standard arguments like in [19, Prop. 10.4 and Prop. 10.6, pp. 34–45] (but with an injective cogenerator of the torsion theory defined by M in $\sigma[M]$ instead of an injective cogenerator of all R-Mod) we get that, more generally, for each monomorphism $Z \to Z'$ of S-Mod, the homomorphism $M \otimes_S Z \to M \otimes_S Z'$ has torsion kernel. Therefore, $M \otimes_S K$ is a torsion module of $\sigma[M]$ and since it is also M-generated,

$M \otimes_S K = 0$. Using again the above argument we see that $M \otimes_S K' = 0$ for every submodule K' of K and this implies (see, e.g. [19, p. 156]) that K is a $\mathscr{F}$-torsion module. Since Y is $\mathscr{F}$-torsionfree we have that $K = 0$. On the other hand, we have also that if $C = \text{Coker } \alpha$, then $M \otimes_S C \simeq \text{Coker } \alpha_* = 0$ and, as before, this implies that C is $\mathscr{F}$-torsion. But since Y is a $\mathscr{F}$-closed submodule of the $\mathscr{F}$-closed module $\text{Hom}_R(M, M \otimes_S Y)$, we must have that C is also $\mathscr{F}$-torsionfree [7, Prop. 5.1], so that $C = 0$. This shows that α is an isomorphism, finishing the proof of (i).

To prove (ii), observe that there are functors

$$M \underset{S}{\otimes} \text{Hom}_R(M, -) : \text{GF}[M] \to \text{CD}[M] \quad \text{and} \quad \bar{-} : \text{CD}[M] \to \text{GF}[M]$$

(the last one defined by setting $X \to \bar{X}$). We claim that these two functors are inverse equivalences of categories. Since, as we have already mentioned, the canonical homomorphism $\beta : M \otimes_S \text{Hom}_R(M, N) \to N$ has torsion kernel and cokernel for each N in $\sigma[M]$, it is clear that if N belongs to $\text{GF}[M]$, then β induces an isomorphism from $M \otimes_S \text{Hom}_R(M, N)$ to N. Also, the above proof shows that $M \otimes_S \text{Hom}_R(M, \bar{X}) \simeq M \otimes_S \text{Hom}_R(M, X) \simeq X$ for each X in $\text{CD}[M]$. Now, if we compose these equivalences with the ones obtained in (i) we see that, since $\beta_* : \text{Hom}_R(M, M \otimes_S \text{Hom}_R(M, N)) \to \text{Hom}_R(M, N)$ is an isomorphism, the functors $\text{Hom}_R(M, -)$ and $M \otimes_S -$ are inverse equivalences between $\text{GF}[M]$ and $(S, \mathscr{F})$-Mod.

Finally, the last part of the statement of Theorem 1.3 is now immediate.

Corollary 1.4. *Let M be a Σ-quasiprojective module. Then* $\text{CD}[M]$ *and* $\text{GF}[M]$ *are Grothendieck categories with projective generators M and $\bar{M}$, respectively.*

The equivalent categories $\text{GF}[M]$ and $\text{CD}[M]$ may be different. Actually, we have:

Proposition 1.5. *Let M be a Σ-quasiprojective module. Then the following conditions are equivalent:*

(i) $\text{GF}[M] \subset \text{CD}[M]$.

(ii) $\text{CD}[M] \subset \text{GF}[M]$.

(iii) *M is M-faithful and the inclusion functor of* $\text{GF}[M]$ *in $\sigma[M]$ is left exact.*

(iv) *M is a self-generator.*

In particular, when any of these conditions holds, $\text{GF}[M] = \text{CD}[M] = \sigma[M]$.

Proof. (i)$\Rightarrow$(ii) Let X be a M-generated module and $p : M^{(I)} \to X$ an epimorphism. Then $\bar{X}$ belongs to $\text{GF}[M]$ and if $\bar{p} : M^{(I)} \to \bar{X}$ is the composition of p with the canonical projection from X to $\bar{X}$, we see that, since by hypothesis M-cod. dim $\bar{X} \geqq 2$, Ker $\bar{p}$ is M-generated. Since $\mathfrak{t}(X)$ is a quotient of Ker $\bar{p}$, this means that $\mathfrak{t}(X) = 0$ and so $X = \bar{X}$, whence X is a module of GF[M].

(ii)⟹(i) Let X be a module of GF[M]. Then by Theorem 1.3 we have that $X \simeq M \otimes_S \mathrm{Hom}_R(M, X)$. Since $M \otimes_S \mathrm{Hom}_R(M, X)$ belongs to CD[M] and hence by (ii) to GF[M], we see that $X \simeq M \otimes_S \mathrm{Hom}_R(M, X)$ also belongs to CD[M].

(ii)⟹(iii) If (ii) (and hence (i)) holds, then clearly GF[M] = CD[M] = $\sigma[M]$ and thus (iii) is clearly verified.

(iii)⟹(iv) As in the proof of (i)⟹(ii) we may see that every quotient module of M is M-faithful. Therefore every submodule of M is the kernel in R-Mod of a morphism of GF[M] and by (iii) it is M-generated.

(iv)⟹(i) Obvious.

Remark. The M-faithfulness of M is not sufficient for the verification of the equivalent conditions of Proposition 1.5. An example of a finitely generated projective module P which is P-faithful but is not a self-generator may be obtained considering the ring R of 3×3 upper triangular matrices over a field and the idempotent $e = (e_{ij}) \in R$, where $e_{11} = e_{33} = 1$, $e_{ij} = 0$ otherwise. Then $P = Re$ is a projective left ideal whose trace in R is $T = ReR$ and clearly P is P-faithful. But if $X \subset P$ is the submodule of all the matrices (a_{ij}) with $a_{12} = a_{21} = a_{22} = a_{31} = a_{32} = a_{33} = 0$, then X is not P-generated for, as it is easily seen, $X_P = TX$ consists of the matrices (b_{ij}) which have $b_{12} = 0$ and zeros in the second and third rows (see also [15]).

2. Self-injective endomorphism rings

Theorem 2.1. *Let M be a left R-module, $S = \mathrm{End}({}_RM)$ and N a M-faithful module. If $\mathrm{Hom}_R(M, N)$ is injective as a left S-module, then N is M-injective. If for every left ideal I of S, the kernel of the canonical homomorphism $M \otimes_S I \to M$ is torsion in $\sigma[M]$, then the converse holds.*

Proof. Assume first that $\mathrm{Hom}_R(M, N)$ is injective as S-module. Let X be a R-submodule of M and I the left ideal of S defined by $I = \mathrm{Hom}_R(M, X)$. The adjunction between the functors $M \otimes_S -$ and $\mathrm{Hom}_R(M, -)$ gives a commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_R(M, N) & \xrightarrow{\simeq} & \mathrm{Hom}_S(S, \mathrm{Hom}_R(M, N)) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_R\left(M \underset{S}{\otimes} I, N\right) & \xrightarrow{\simeq} & \mathrm{Hom}_S(I, \mathrm{Hom}_R(M, N)) \end{array}$$

in which the horizontal arrows are isomorphisms and the vertical arrow on the right is an epimorphism. Thus the vertical arrow on the left is also an epimorphism. This means that every homomorphism $g: M \otimes_S I \to N$ factors through the canonical morphism $u: M \otimes_S I \to M$ and, since $\operatorname{Im} u = X_M$, each homomorphism $h: X_M \to N$ extends to M. Now, if $f: X \to N$ is a homomorphism, we see that there exists $t: M \to N$ such that $t|_{X_M} = f|_{X_M}$. Then $t|_X - f$ factors through the projection $X \to X/X_M$, and since X/X_M is a torsion module and N is M-faithful we get that $t|_X = f$. Then N is M-injective.

Conversely, assume that N is M-injective and let I be a left ideal of S. Since the kernel of the canonical morphism $u: M \otimes_S I \to M$ is a torsion module of $\sigma[M]$ and N is M-faithful, we see that, if $f: M \otimes_S I \to N$, then f factors through $M \otimes_S I \to \operatorname{Im} u = MI \subset M$. Since N is M-injective, f can be factored through u, and so in the above diagram, the vertical arrow on the left is an epimorphism. Thus the vertical arrow on the right is also an epimorphism and this implies that $\operatorname{Hom}_R(M, N)$ is an injective S-module.

Remarks. The same proof of Theorem 2.1 shows that the result is still valid if we take N a module (not necessarily of $\sigma[M]$) such that its largest M-subgenerated submodule is M-faithful. On the other hand, a slight modification of the proof of Theorem 2.1 shows that, for any left R-module N, if $\operatorname{Hom}_R(M, N)$ is injective, then every homomorphism from a M-generated submodule of M to N has an extension to M. If, moreover, M_S is flat, then the converse holds.

When M_S is flat and N is M-faithful, then Theorem 2.1 says that $\operatorname{Hom}_R(M, N)$ is injective if and only if N is M-injective. Similarly, we have for Σ-quasiprojective modules:

Corollary 2.2. *Let M be a Σ-quasiprojective module and N a M-faithful module. Then* $\operatorname{Hom}_R(M, N)$ *is an injective S-module if and only if N is M-injective. In particular, S is left self-injective if and only if $\bar{M}$ is quasiinjective.*

Proof. As we have seen in the proof of Theorem 1.3, the canonical homomorphism $M \otimes_S I \to M$ has torsion kernel in $\sigma[M]$ for every left ideal I of S and so Theorem 2.1 applies. For the last part note that, if M is Σ-quasiprojective, then $\bar{M}$ is also Σ-quasiprojective and $\bar{M}$-faithful and S is isomorphic to $\operatorname{End}({}_R\bar{M})$, so that the result follows from the first part.

We will denote by $E(X)$ the injective envelope of a R-module X.

COROLLARY 2.3. *Let P be a projective module with trace T and $S =$ End$({}_RP)$. Then, the following conditions are equivalent*:

(i) *S is left self-injective.*

(ii) $\bar{P} = TE(\bar{P})$.

(iii) $\mathrm{Hom}_R(T \otimes_R T, P)$ *is an injective left R-module.*

PROOF. Clearly, we have that, in this case, $E(\bar{P})_P = E(\bar{P})_{\bar{P}} = TE(\bar{P})$. Thus (ii) is equivalent to $\bar{P}$ being quasiinjective (see [1, Prop. 2.5]) and so the equivalence between (i) and (ii) follows from Corollary 2.2.

To show that (i) is equivalent to (iii), consider the Gabriel topology of R defined by $\mathscr{F}_P = \{I \subset {}_RR \mid T \subset I\}$ and the Gabriel topology of S, $\mathscr{F}^P = \{J \subset {}_SS \mid T' \subset J\}$ where T' is the trace of P in S, i.e., the image of the canonical homomorphism $P^* \otimes_R P \to S$). Then there is an equivalence of categories between $(R, \mathscr{F}_P)$-Mod and $(S, \mathscr{F}^P)$-Mod in which $P_{\mathscr{F}_P}$ corresponds to S (which is a $\mathscr{F}^P$-closed module) (see [4]). Thus S is left self-injective if and only if S is injective in $(S, \mathscr{F}^P)$-Mod if and only if $P_{\mathscr{F}_P}$ is injective in $(R, \mathscr{F}_P)$-Mod if and only if $P_{\mathscr{F}_P}$ is injective in R-Mod [19, Prop. 1.7, p. 215]. But, using [19, Example 3, p. 260] we see that $P_{\mathscr{F}_P} \simeq \mathrm{Hom}_R(T \otimes_R T, P)$ from which the result follows.

COROLLARY 2.4. *Let R be a commutative ring, M a finitely generated quasiprojective module and $S =$ End$({}_RM)$. The following conditions are equivalent*:

(i) *S is a left self-injective ring.*

(ii) *M is quasiinjective.*

(iii) $R/\mathrm{Ann}_R(M)$ *is a self-injective ring.*

PROOF. From [9, Folgerung 2.34] it follows that M is a self-generator and in fact it is a projective generator of $(R/\mathrm{Ann}_R(M))$-Mod, so that S and $R/\mathrm{Ann}_R(M)$ are Morita-equivalent rings. Thus the result is clear.

PROPOSITION 2.5. *Let P be a projective module and N a module of P-*cod. dim $\geqq 2$. *If* $\mathrm{Hom}_R(P, N)$ *is an injective S-module, then* $\mathrm{Ext}_R(X, N) = 0$ *for every X such that P-*cod. dim $X \geqq 3$. *If furthermore P_S is flat, then the converse holds. In particular, if P_S is flat, then S is left self-injective if and only if* $\mathrm{Ext}_R(X, P) = 0$ *for every X such that P-*cod. dim $X \geqq 3$.

PROOF. According to Theorem 1.3, there is an equivalence of categories between CD[P] and $(S, \mathscr{F})$-Mod (where $\mathscr{F} = \{I \subset {}_SS \mid PI = P\}$), which is induced by $\mathrm{Hom}_R(P, -)$. If $\mathrm{Hom}_R(P, N)$ is an injective S-module, then it is also an injective object of $(S, \mathscr{F})$-Mod and thus N is an injective object of CD[P]. Let X be a module of P-codominant dimension $\geqq 3$ and $p: P^{(I)} \to X$ an epimorphism with $K = \operatorname{Ker} p$. Then we get an exact sequence in R-Mod:

$$\operatorname{Hom}_R(P^{(I)}, N) \to \operatorname{Hom}_R(K, N) \xrightarrow{v} \operatorname{Ext}_R(X, N) \to \operatorname{Ext}_R(P^{(I)}, N) = 0,$$

in which, since the inclusion of K in $P^{(I)}$ is clearly a monomorphism of CD[P], $v = 0$ and hence $\operatorname{Ext}_R(X, N) = 0$.

For the converse note that, according to the proof of Theorem 2.1, $\operatorname{Hom}_R(P, N)$ is S-injective if and only if, for every left ideal I of S, every $f: P \otimes_S I \to N$ factors through the canonical homomorphism $P \otimes_S I \to P$. Since P_S is flat, this is a monomorphism and also it is clear that $P \otimes_S I$ has P-codominant dimension $\geqq 2$, so that the condition of the Proposition holds.

We now study when the endomorphism ring of a Σ-quasiprojective module is quasi-Frobenius (QF).

PROPOSITION 2.6. *Let M be a Σ-quasiprojective module and $S = \operatorname{End}(_RM)$. The following conditions are equivalent*:

(i) *S is* QF.

(ii) *M is finitely generated and each M-injective module of* GF[M] *is isomorphic to a direct summand of $\bar{M}^{(J)}$ for some set J.*

(iii) *M is finitely generated and every direct summand of $\bar{M}^{(J)}$ is M-injective.*

(iv) *M is finitely generated and each M-generated M-faithful module embeds in $\bar{M}^{(J)}$ for some set J.*

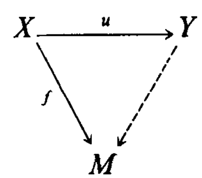
PROOF. If S is a QF ring, then $_SS$ is a cogenerator of S-Mod and hence, if we consider the Gabriel topology $\mathscr{F} = \{I \subset {_SS} \mid MI = M\}$, we see that, as S is $\mathscr{F}$-torsionfree, every left S-module is $\mathscr{F}$-torsionfree. Thus $\mathscr{F}$ is trivial and hence M is finitely generated by Theorem 1.3. Therefore we may assume through all the proof that M is finitely generated and $\operatorname{Hom}_R(M, -)$: GF[$M$] $\to S$-Mod is an equivalence. To see that (i) is equivalent to (ii), note that S is QF if and only if every injective S-module is projective, that is, if and only if every injective object of GF[M] is projective in GF[M]. Since $\bar{M}$ is a generator of GF[M] and the monomorphisms of GF[M] are just the injective homomorphisms, we see that an object of GF[M] is injective if and only if it is $\bar{M}$-injective [1, 1.14] and this is clearly equivalent to being M-injective. As $\bar{M}$ is a projective generator of GF[M], the projective objects of GF[M] are precisely the direct summands (in GF[M] and also in R-Mod) of modules of the form $\bar{M}^{(J)}$.

The equivalence between (i) and (iii) follows in a similar way (using the fact that S is QF if and only if every projective S-module is injective) and the equivalence between (i) and (iv) follows from the fact that S is QF if and only if every left R-module embeds in a free module.

If M is a M-faithful but not necessarily Σ-quasiprojective R-module, then it is still possible to use Theorem 2.1 to study when $\mathrm{End}({}_RM)$ is left self-injective. Let M be any left R-module and $\mathcal{T}$ the torsion class of $\sigma[M]$ defined in the beginning of §1. The full subcategory of $\sigma[M]$ determined by $\mathcal{T}$ is a localizing subcategory [16, Theorem 6.3, p. 186] and thus there exists an associated quotient category which we denote by $\mathscr{C}[M]$, with canonical functor $\mathrm{T}: \sigma[M] \to \mathscr{C}[M]$. T is an exact functor and has a right adjoint $\mathrm{S}: \mathscr{C}[M] \to \sigma[M]$ which is full and faithful [16, pp. 172–176]. $\mathscr{C}[M]$ is a Grothendieck category [16, Corollary 6.2, p. 186].

THEOREM 2.7. *Let M be a M-faithful module and $S = \mathrm{End}({}_RM)$. Then S is left self-injective if and only if ${}_RM$ is quasiinjective.*

PROOF. If S is left self-injective, then M is quasiinjective by Theorem 2.1. Then M is clearly $\mathcal{T}$-injective, that is, every diagram in $\sigma[M]$:



with u a monomorphism such that $\mathrm{Coker}\, u \in \mathcal{T}$ can be completed. But $\mathscr{C}[M]$ can be identified with the full subcategory of $\sigma[M]$ determined by the M-faithful $\mathcal{T}$-injective modules (see [16, p. 177] and [20]) and hence in this case $M = \mathrm{T}(M)$ is an object of $\mathscr{C}[M]$. As in [1, Prop. 8.6], we may see that M is a generator of $\mathscr{C}[M]$. By the Gabriel–Popescu theorem [16, Theorem 14.2, p. 248], the functor $\mathrm{Hom}_R(M, -): \mathscr{C}[M] \to S\text{-Mod}$ has an exact left adjoint. Since this functor factors in the form

$$\mathscr{C}[M] \xrightarrow{\mathrm{S}} \sigma[M] \xrightarrow{\mathrm{Hom}_R(M,-)} S\text{-Mod},$$

its left adjoint can be obtained by composing the left adjoint $M \otimes_S -$ of $\mathrm{Hom}_R(M, -)$ with the left adjoint T of S. Thus the functor $\mathrm{T}(M \otimes_S -): S\text{-Mod} \to \mathscr{C}[M]$ is exact and this implies that if I is a left ideal of S, then $\mathrm{T}(M \otimes_S I \to M)$ is a monomorphism of $\mathscr{C}[M]$. By [16, Lemma 3.5,

p. 170], this is equivalent to the fact that $\mathrm{Ker}(M \otimes_S I \to M)$ is a torsion module of $\sigma[M]$ and hence Theorem 2.1 completes the proof.

REMARKS. When M is a Σ-quasiprojective module, $\mathrm{Hom}_R(M, -)$ induces an equivalence between $\mathscr{C}[M]$ and $(S, \mathscr{F})$-Mod (with $\mathscr{F} = \{I \subset {}_S S \mid MI = M\}$) [5, Theorem 1.3]. This also follows from the fact that, as it can be readily shown, $\mathrm{T}: \sigma[M] \to \mathscr{C}[M]$ induces by restriction an equivalence between GF[M] and $\mathscr{C}[M]$, with inverse given by $X \to X_M$. Nevertheless, unlike what happens in the hypotheses of Theorem 2.7, a M-faithful module M may not belong to $\mathscr{C}[M]$ (even in case M is also assumed to be projective). For instance, if X and P are the modules considered in the remark following Proposition 1.5 and $f: X_P \to P$ is the homomorphism defined by $f((b_{ij})) = (c_{ij})$ with $c_{11} = b_{13}$, $c_{13} = b_{11}$ and $c_{ij} = 0$ otherwise, then f has no extension to X. Also (for M Σ-quasiprojective) CD[M] may be different from $\mathscr{C}[M]$ and in fact CD[M] $= \mathscr{C}[M]$ if and only if M is a self-generator.

COROLLARY 2.8. *Let U be a generator of R-*Mod *and $M = U^{(I)}$ an infinite direct sum of copies of U. Then $S =$* End$({}_R M)$ *is left self-injective if and only if R is a quasi-Frobenius ring and ${}_R U$ is projective.*

PROOF. Clearly M is M-faithful and so S is left self-injective if and only if M is quasiinjective. Since M is a generator of R-Mod, this is in turn equivalent to M being an injective R-module [1, Coroll. 1.10]. By [2, Theorem 25.1] we have that $U^{(J)}$ is injective for every set J and therefore every projective left R-module is injective, so that R is QF. Moreover U, being injective, is also projective. The converse is clear.

We recall that if M is a quasiinjective module, then a submodule X of M is said to be (finitely) closed if M/X is (finitely) cogenerated by M [1]. Also, we will say that a submodule L of a module N of $\sigma[M]$ is saturated in N in case the quotient module N/L is M-faithful. Given a submodule X of N, we denote by X^c the smallest saturated submodule of N containing X, so that $X^c/X = \mathrm{t}(N/X)$. The ascending (descending) chain condition will be denoted by ACC (DCC).

COROLLARY 2.9. *Let M be a M-faithful module and $S =$* End$({}_R M)$. *The following conditions are equivalent*:

(i) *S is a* QF *ring.*
(ii) *M is quasiinjective and satisfies the* DCC *on M-generated submodules.*
(iii) *M is quasiinjective and satisfies the* DCC *on (finitely) closed submodules.*

(iv) *M is quasiinjective and satisfies the* ACC *on M-generated submodules.*

PROOF. (i)$\Rightarrow$(ii) Since S is left self-injective we have, by Theorem 2.7, that M is quasiinjective. On the other hand, S satisfies the DCC on left ideals of the form $\mathrm{Hom}_R(M, X)$, with $X \subset M$ (because S is left artinian). These left ideals correspond by an order-preserving bijection to the M-generated submodules of M and hence M has DCC on M-generated submodules.

(ii)$\Rightarrow$(iii) Observe that, since M is M-faithful, any (finitely) closed submodule X of M is saturated (because M/X is cogenerated by the M-faithful module M). But it is clear that $Y \to Y_M$ defines an order-preserving injection, with left inverse $Z \to Z^c$, from the set of saturated submodules of M to the set of M-generated submodules of M. Then, the DCC on M-generated submodules implies the DCC on (finitely) closed submodules.

(iii)$\Rightarrow$(i) This follows from Theorem 2.7 and [1, Coroll. 4.3], because in this case S is left self-injective and right noetherian.

(i)$\Rightarrow$(iv) This is similar to (i)$\Rightarrow$(ii).

(iv)$\Rightarrow$(i) By Theorem 2.7, S is left self-injective. As in the proof of (ii)$\Rightarrow$(iii), we see that it follows from (iv) that M satisfies the ACC on saturated submodules; and, as in the proof of Theorem 2.7, we have, by the Gabriel–Popescu Theorem, an equivalence of categories $\mathrm{Hom}_R(M, -) : \mathscr{C}[M] \to (S, \mathscr{F})$-Mod, with $\mathscr{F}$ the left Gabriel topology of S consisting of the left ideals I such that $\mathbf{T}(M \otimes_S S/I) = 0$, or, equivalently, M/MI is a torsion module of $\sigma[M]$. As in [5, Prop. 1.1], we have that there exists an isomorphism between the lattice of subobjects of M in $\mathscr{C}[M]$ and the lattice of saturated submodules of M, so that in this case, M is a noetherian object of $\mathscr{C}[M]$ and, since S corresponds to M in the equivalence, S is a noetherian object of $(S, \mathscr{F})$-Mod. This means that S has the ACC on $\mathscr{F}$-saturated left ideals [19, Coroll. 4.4, p. 208]. Since every left annihilator of S is $\mathscr{F}$-saturated (because S is $\mathscr{F}$-torsionfree) we see that S satisfies the ACC on left annihilators, which implies that S is QF [19, Theorem 3.5, p. 277].

REMARK. Note that from the proof of Corollary 2.9 it follows that, for a M-faithful module M, S is QF if and only if M is quasiinjective and satisfies the ACC (or the DCC) on saturated submodules.

COROLLARY 2.10. *Let M be a Σ-quasiprojective module and $S = \mathrm{End}({}_RM)$. Then S is* QF *if and only if M is a quasiinjective module with* ACC (*or* DCC) *on M-generated submodules.*

We recall that if P is a projective R-module with trace T in R, then a left R-

module X is called T-accessible if $TX = X$, i.e., if X is P-generated; and a module N is T-noetherian (T-artinian) if it satisfies the ACC (DCC) on T-accessible submodules [1]. When P is finitely generated, P^* is a finitely generated projective right R-module and its trace is also T [1, Lemma 8.10]. We will denote by $\overline{P^*}$ the quotient of P^* modulo its torsion submodule in $\sigma[P^*]$.

COROLLARY 2.11. *Let P be a projective module and $S = \mathrm{End}({}_RP)$. The following conditions are equivalent*:

(i) *S is* QF.

(ii) *P is T-artinian (T-noetherian) and $\bar{P}$ is quasiinjective.*

(iii) *P is T-artinian (T-noetherian) and $\overline{P^*}$ is quasiinjective.*

PROOF. The equivalence of (i) and (ii) follows from Corollary 2.9. The equivalence between (i) and (iii) follows from the fact that S is isomorphic to $\mathrm{End}(\overline{P_R^*})$ and hence, by Theorem 2.7, S is right self-injective if and only if $\overline{P^*}$ is quasiinjective.

3. PF endomorphism rings

We recall that a ring R is said to be left Kasch [19] (or right S-ring [11]) in case it has no proper dense left ideals. This amounts to saying that each simple left R-module is isomorphic to a minimal left ideal, or, equivalently, that $E({}_RR)$ is a cogenerator of R-Mod. On the other hand, a module M is called a RZ-module [20] if every simple quotient of M is cogenerated by M. We have:

THEOREM 3.1. *Let M be a Σ-quasiprojective module, $S = \mathrm{End}({}_RM)$. The following conditions are equivalent*:

(i) *S is a left Kasch ring.*

(ii) *$\bar{M}$ is a finitely generated RZ-module.*

(iii) *M is finitely generated and for every maximal M-generated submodule X of M there is a homomorphism from M/X to M with torsion kernel.*

(iv) *M is finitely generated and every M-faithful module is cogenerated by $E(\bar{M})$.*

PROOF. If S is left Kasch, then $E(S)$ is a cogenerator of S-Mod. If $\mathscr{F} = \{I \subset {}_SS \mid MI = M\}$, then, since S is $\mathscr{F}$-torsionfree, so is $E(S)$. Therefore, $\mathscr{F} = \{S\}$ and M is finitely generated by Theorem 1.3. On the other hand, $\mathbf{t}(M)$ is always a superfluous submodule of M, because if X is a submodule of M such that $X + \mathbf{t}(M) = M$, then $X^c + \mathbf{t}(M) = X^c = M$ and, since M is Σ-quasiprojective, we get that $X = M$. From this, it follows easily that $\bar{M}$ is finitely generated if and only if M is. Thus we can make the general assumption that M is finitely generated and hence, by Theorem 1.3, we have equivalences of categories $\mathrm{Hom}_R(M, -): \mathrm{GF}[M] \to S\text{-Mod}$ and $\mathrm{Hom}_R(M, -): \mathrm{CD}[M] \to S\text{-Mod}$.

To see that (i) and (ii) are equivalent observe that, by the above equivalence, S is left Kasch if and only if every simple object of $\mathrm{GF}[M]$ is isomorphic to a subobject of $\bar{M}$. It is easy to see that the simple objects of $\mathrm{GF}[M]$ are precisely the simple quotients of $\bar{M}$, and hence S is left Kasch if and only if (ii) holds.

The equivalence between (i) and (iii) follows in a similar way. The equivalence of categories $\mathrm{Hom}_R(M, -): \mathrm{CD}[M] \to S\text{-Mod}$ shows that (i) is equivalent to every simple object of $\mathrm{CD}[M]$ being isomorphic to a subobject of M in $\mathrm{CD}[M]$. Clearly, the simple objects of $\mathrm{CD}[M]$ are the quotients of M modulo maximal M-generated submodules and it is easy to see (e.g., by using the equivalence with S-Mod) that the monomorphisms of $\mathrm{CD}[M]$ are just homomorphisms whose kernel is a torsion module of $\sigma[M]$, from which the equivalence follows.

Finally, the equivalence between (i) and (iv) is obtained by using the equivalence $\mathrm{Hom}_R(M, -): \mathrm{GF}[M] \to S\text{-Mod}$ and the fact that S is left Kasch if and only if $E(S)$ is a cogenerator of S-Mod. This means that (i) is equivalent to the injective envelope of $\bar{M}$ in $\mathrm{GF}[M]$ being a cogenerator of $\mathrm{GF}[M]$. On the other hand, if $E(\bar{M})$ denotes the injective envelope of $\bar{M}$ in R-Mod, then it is clear that $E(\bar{M})_M$ is an injective object of $\sigma[M]$ and hence of $\mathrm{GF}[M]$ and thus we see that $E_{\mathrm{GF}[M]}(\bar{M}) = E(\bar{M})_M$. Since products in $\mathrm{GF}[M]$ are calculated by taking the M-generated part of products in R-Mod, it is clear that if (iv) holds, then $E(\bar{M})_M$ is a cogenerator of $\mathrm{GF}[M]$ and hence S is left Kasch. Conversely, assume that $E(\bar{M})_M$ is a cogenerator of $\mathrm{GF}[M]$ and let X be a M-faithful module. Then X_M is an object of $\mathrm{GF}[M]$ and so it is clear that there is a monomorphism from X_M to $E(M)^J$, for some set J. But, obviously, X_M is an essential submodule of X and the injectivity of $E(M)^J$ gives a monomorphism from X to $E(M)^J$, which completes the proof.

The following result extends [13, Théorème 2.1].

Proposition 3.2. *Let M be a Σ-quasiprojective module and N a module such that M-*cod. dim $N \geqq 2$ *and M is N-generated. Let $S = \mathrm{End}(_RM)$ and $S' = \mathrm{End}(_RN)$. Then the following conditions are equivalent*:

(i) *S is left Kasch and N is a finitely generated quasiprojective module.*

(ii) *S' is left Kasch and M is finitely generated.*

Proof. (i)$\Rightarrow$(ii) By Theorem 3.1 M is finitely generated and by Theorem

1.3 there is an equivalence between CD[M] and S-Mod. Since M is N-generated and coproducts in CD[M] coincide with coproducts in R-Mod, we see that N is a generator of CD[M]. Since by (i) N is a Σ-quasiprojective module, it is a projective object of $\sigma[M]$ and hence of CD[M]. On the other hand, since N is finitely generated there exists an epimorphism $M^n \to N$ (in R-Mod and in CD[M]). Since M is clearly a finitely generated object of CD[M], we see that N is actually a finitely generated projective generator of CD[M] and hence, by the Gabriel–Popescu theorem, CD[M] is equivalent to S'-Mod. Therefore S and S' are Morita-equivalent rings and so S' is left Kasch.

(ii)$\Rightarrow$(i) As before, we have that N is a generator of CD[M]. By the Gabriel–Popescu theorem, there is an equivalence, induced by $\mathrm{Hom}_R(N, -)$, between CD[M] and S'-Mod/$\mathscr{F}'$, where $\mathscr{F}'$ is a left Gabriel topology of S' such that S' is $\mathscr{F}'$-torsionfree. Since S' is left Kasch, the $\mathscr{F}'$-torsionfree module $E(S')$ cogenerates S'-Mod and hence $\mathscr{F}'$ is trivial, so that $\mathrm{Hom}_R(N, -)$: CD[M]$\to S'$-Mod is an equivalence. By Theorem 1.3, CD[M] is also equivalent to S-Mod and hence S is a left Kasch ring. On the other hand, since N corresponds to S' in the equivalence between CD[M] and S'-Mod, N is a finitely generated projective object of CD[M]. Thus N is isomorphic to a direct summand of some M^n and hence it is a finitely generated quasiprojective module.

REMARK. If in Proposition 3.2 we take $M = R$, we get [13, Théorème 2.1]. Also, it follows from Proposition 3.2 that if M is a finitely generated quasiprojective module, then $\mathrm{End}(M^{(J)})$ is left Kasch if and only if $\mathrm{End}({}_RM)$ is left Kasch and J is finite.

We will make use of the following lemma.

LEMMA 3.3. *Let M be a M-faithful quasiprojective module and $S =$ $\mathrm{End}({}_RM)$. Then ${}_SS$ is finitely cogenerated if and only if ${}_RM$ is finitely cogenerated.*

PROOF. Assume that ${}_RM$ is finitely cogenerated, so that the socle $\mathrm{Soc}({}_RM)$ is a finitely generated essential submodule of M. By [1, Coroll. 4.10], the finitely M-generated submodules N of M correspond by an order-preserving bijection given by $N \to \mathrm{Hom}_R(M, N)$ (with inverse $X \to MX$) to the finitely generated left ideals of S. Since M is M-faithful, the minimal finitely M-generated submodules of M are precisely the simple submodules of M and these correspond to the minimal left ideals of S. Thus it is clear that if $\mathrm{Soc}({}_RM) =$ $\bigoplus_1^n C_j$ with each C_j simple, then $\mathrm{Soc}({}_SS) = \bigoplus_1^n \mathrm{Hom}(M, C_j)$ and hence $\mathrm{Soc}({}_SS)$

is finitely generated. On the other hand, if I is a finitely generated left ideal of S, then MI is a finitely M-generated submodule of M which contains (since $\mathrm{Soc}({}_RM)$ is essential in M) a simple submodule C and so $I = \mathrm{Hom}_R(M, MI)$ contains the minimal left ideal of S $\mathrm{Hom}_R(M, C)$. Thus $\mathrm{Soc}({}_SS)$ is essential and hence ${}_SS$ is finitely cogenerated. The converse is similar.

We recall that a ring R is called left pseudo-Frobenius (PF) if R is an injective cogenerator of R-Mod, or, equivalently, R is left self-injective and ${}_RR$ is finitely cogenerated [10]. In order to study when the endomorphism ring of a quasiprojective module is left PF we first show:

Proposition 3.4. *Let M be a M-faithful quasiprojective module. If S is a left* PF *ring, then M is finitely generated.*

Proof. By [10, Coroll. 11.4.3], $S = \bigoplus_1^n Se_i$ where $\{e_i \mid i = 1, \dots, n\}$ is a set of orthogonal idempotents and e_iSe_i is a local ring, for each i. Taking $M_i = Me_i$ we have that $M = \bigoplus_1^n M_i$ and $\mathrm{End}(M_i) \simeq e_iSe_i$ is local; therefore, each M_i is a indecomposable module. Since S is finitely cogenerated, so is M (Lemma 3.3) and thus each M_i has a simple submodule T_i, which, being M-faithful, is isomorphic to a quotient of M.

On the other hand, M is quasiinjective by Theorem 2.1, and thus each M_i is M-injective [1, Prop. 1.4], that is, M_i is an injective object of $\sigma[M]$. Since the injective envelope of T_i in this category is $E(T_i)_M$ and M_i is indecomposable we have that $M_i = E(T_i)_M$.

Suppose now that $\{T_1, \dots, T_r\}$ is a set of representatives of the isomorphism classes of $T_1, \dots, T_n$. Accordingly, $\{M_1, \dots, M_r\}$ is a set of representatives of the isomorphism classes of $M_1, \dots, M_n$, and since $\mathrm{Hom}_R(M, T_i) \neq 0$, we deduce that there exists some j of $\{1, \dots, r\}$ such that $\mathrm{Hom}_R(M_j, T_i) \neq 0$, and thus T_i is a simple quotient of M_j. On the other hand, we may see, as in the proof of [2, 17.19, (c)$\Rightarrow$(a)] that each proper submodule of any M_j is superfluous and so M_j has a unique maximal submodule, which contains every proper submodule, that is, M_j is a local module. If we choose i' in $\{1, \dots, r\}$ such that $i' \neq i$, then we have $\mathrm{Hom}_R(M_{j'}, T_{i'}) \neq 0$ for some j' in $\{1, \dots, r\}$, and $j' \neq j$, for $j' = j$ would imply that M_j has two nonisomorphic simple quotients, which contradicts the fact that M_j is local. Therefore, we get for each $i \in \{1, \dots, r\}$ an index $j(i) \in \{1, \dots, r\}$ such that $M_{j(i)}$ is a local module and, since all the $j(i)$ are different, we see that $M = \bigoplus_1^n M_i$ is a finite direct sum of local modules (which are cyclic) and hence it is finitely generated.

In [17] Rutter called PF modules to the finitely generated projective and injective RZ-modules, and showed that if P is a PF module, then $S = \mathrm{End}(_RP)$ is a left PF ring. Also, in [14] it is shown that if P is a finitely cogenerated projective and injective RZ-module, then S is left PF and, in fact, these modules are precisely the PF modules. We are going to extend these results, giving necessary and sufficient conditions for the endomorphism ring of a Σ-quasiprojective module to be left PF.

THEOREM 3.5. *Let M be a M-faithful quasiprojective module. Then the following conditions are equivalent*:

(i) *S is left* PF.

(ii) *M is a finitely cogenerated quasiinjective module.*

(iii) *M is a finitely generated quasiinjective RZ-module.*

(iv) *M is a finitely generated quasiinjective module, which is semisimple modulo its radical and has essential socle.*

(v) *M is finitely generated and every M-generated (M-generated M-faithful) module that cogenerates M generates M.*

PROOF. (i)$\Leftrightarrow$(ii) This follows from Theorem 2.7 and Lemma 3.3.

To prove the equivalence of (i) with the remaining conditions observe that, by Proposition 3.4, it follows from (i) that M is a finitely generated R-module and thus in what follows we will assume that M is finitely generated. Then M is in fact Σ-quasiprojective and we have as usual an equivalence $\mathrm{Hom}_R(M, -)$: $\mathrm{GF}[M] \to S$-Mod.

(i)$\Leftrightarrow$(iii) Since R is left PF if and only if it is left self-injective and left Kasch, this follows from Theorems 2.7 and 3.1.

(i)$\Leftrightarrow$(iv) From the fact that $\mathrm{Hom}_R(M, -)$ defines an equivalence between $\mathrm{GF}[M]$ and S-Mod we infer that S has essential left socle if and only if M has essential socle in the category $\mathrm{GF}[M]$. Since the simple objects of $\mathrm{GF}[M]$ are clearly the M-faithful simple modules (i.e., the simple quotients of M), we see that the socle of M in $\mathrm{GF}[M]$ coincides with its socle in R-Mod, and hence S has essential left socle if and only if M has essential socle. By [10, Theorem 12.5.2] S is left PF if and only if it is a left self-injective semilocal ring with essential left socle. Thus, by using Theorem 2.7 and the foregoing remarks, we see that to prove the equivalence of (i) and (iv) it will be enough to show that S is semilocal if and only if M is semisimple modulo its radical. This is a consequence of the facts that, as it can be readily shown, $\mathrm{Rad}(M) = (\mathrm{Rad}_{\mathrm{GF}[M]}(M))^c$ and the cokernel of $\mathrm{Rad}_{\mathrm{GF}[M]}(M) \to M$ in $\mathrm{GF}[M]$ is precisely $M/\mathrm{Rad}(M)$ (see also [5, Prop. 3.7]).

(i)$\Leftrightarrow$(v) As it is well known, a ring S is left PF if and only if every faithful left

S-module is a generator of S-Mod, so that all we have to show is that in this case, this condition for $S = \mathrm{End}({}_RM)$ is equivalent to (v). Through the category equivalence between GF[M] and S-Mod we get that S is left PF if and only if every module of GF[M] which cogenerates M in GF[M] also generates M in GF[M]. Since the inclusion functor from GF[M] to R-Mod clearly preserves coproducts, monomorphisms and epimorphisms, we see that this condition amounts to saying that every M-generated M-faithful module which cogenerates M (or, equivalently, cogenerates M in GF[M], because the product in GF[M] of a family $\{X_i\}$ is given by $(\Pi X_i)_M$) generates M. Now, if X is a M-generated module which cogenerates M, then, since M is M-faithful and the torsion functor of $\sigma[M]$ preserves products (as a consequence of the fact that the torsion class $\mathscr{T}$ consists of the modules N of $\sigma[M]$ such that $\mathrm{Hom}_R(M, N) = 0$) we see that also $\bar{X}$ cogenerates M. Since $\bar{X}$ is an object of GF[M] it follows that $\bar{X}$ generates M, and thus so does X, which completes the proof.

Corollary 3.6. *Let M be a Σ-quasiprojective module and $S = \mathrm{End}({}_RM)$. The following conditions are equivalent:*

(i) *S is left* PF.

(ii) *$\bar{M}$ is a finitely cogenerated quasiinjective module.*

(iii) *$\bar{M}$ is a finitely generated quasiinjective RZ-module.*

(iv) *$\bar{M}$ is a finitely generated quasiinjective module which is semisimple modulo its radical and has essential socle.*

(v) *M is finitely generated and every M-generated module that cogenerates $\bar{M}$ generates $\bar{M}$.*

Remarks. In Theorem 3.5 and Corollary 3.6 we cannot leave out the hypotheses of M (or $\bar{M}$) being finitely generated in (iii), (iv) and (v) and finitely cogenerated in (ii). For instance, if M is a nonfinitely generated semisimple module, then all of the conditions (ii)–(v) (without the finiteness hypotheses) hold but (i) does not hold.

If P is a projective module with trace T, then, as we have already observed $\bar{P}$ is quasiinjective if and only if $\bar{P} = TE(\bar{P})$. Thus we see that $S = \mathrm{End}({}_RP)$ is left PF if and only if $\bar{P} = TE(\bar{P})$ and $\bar{P}$ is either a finitely cogenerated module or a finitely generated RZ-module.

We recall that if P is a projective module with trace T, a module X is called T-faithful if for any x of X, $Tx = 0$ implies $x = 0$. This is clearly equivalent to X being a P-distinguished module, and also to X_P being a P-faithful module.

PROPOSITION 3.7. *Let P be a finitely generated projective module. Then P is a* PF *module if and only if it is T-faithful and every module which cogenerates P generates P.*

PROOF. Assume that P is PF. Then P is finitely cogenerated [14, p. 695] and thus $P = \bar{P}$ [14, Lemma 1]. Therefore, each module X which cogenerates P does cogenerate it finitely and, since P is injective, X generates P.

Conversely, if P is T-faithful and every module which cogenerates P generates P, then it follows from Corollary 3.6 that P is a RZ-module. Since P is cogenerated by $E(P)$, it is also (finitely) generated by $E(P)$, which means that P is injective and completes the proof.

COROLLARY 3.8. [14]. *If P is either a* PF *module or a finitely cogenerated projective and injective RZ-module, then* $\mathrm{End}({}_RP)$ *is a left* PF *ring.*

PROOF. If P is PF, the result follows from Proposition 3.7 and Corollary 3.6. In the other case, we see as in the proof of Proposition 3.7 that P is P-faithful and so the result follows from Corollary 3.6.

REMARKS. Proposition 3.7 shows that PF modules in the sense of Page [15] are the same as PF modules in the sense of Rutter [17] and thus it is stronger than [15, Proposition 3].

If P is a finitely generated projective module, the condition of being a PF module is sufficient but not necessary for the endomorphism ring $S = \mathrm{End}({}_RP)$ to be left PF. For instance, if R is the ring of 2×2 upper triangular matrices over a field k and e is the idempotent of R given by $e = (e_{ij})$, with $e_{11} = 1$, $e_{ij} = 0$ otherwise, then $P = Re$ is a T-faithful projective simple R-module and $\mathrm{End}({}_RP) \simeq k$. Nevertheless P is not an injective R-module and thus it is not PF.

On the other hand, there are finitely generated (and finitely cogenerated) projective and injective modules P such that $\mathrm{End}({}_RP)$ is left PF but P is not a RZ-module (and hence P is not T-faithful), which shows, among other things, that we cannot substitute M for $\bar{M}$ in Corollary 3.6. A simple example of this is obtained by considering the same ring as above and $P' = Re'$, with $e' = (e'_{ij})$ given by $e'_{22} = 1$, $e'_{ij} = 0$ otherwise. Then $\mathrm{End}(Re') \simeq e'Re' \simeq k$ is PF and P' is a finitely cogenerated injective module which is not a RZ-module ($\bar{P}'$ is a simple module which is not cogenerated by P'). Nevertheless, if P is a T-faithful projective and injective module, then $\mathrm{End}({}_RP)$ is left PF if and only if P is PF (this follows from Corollary 3.6; compare with [15, Coroll. 5]).

We will say that R is a PF ring if R is both left and right PF (these rings are also called rings with perfect duality [10] or cogenerator rings).

COROLLARY 3.9. *Let P be a finitely generated projective module with trace T. The following conditions are equivalent*:

(i) S *is* PF.

(ii) $\bar{P} = TE(\bar{P})$, $\overline{P^*} = E(\overline{P^*})T$ *and* ${}_R\bar{P}$ *and* $\overline{P^*_R}$ *are RZ-modules*.

(iii) $\bar{P} = TE(\bar{P})$, $\overline{P^*} = E(\overline{P^*})T$ *and* ${}_R\bar{P}$ *and* $\overline{P^*_R}$ *are finitely cogenerated modules*.

PROOF. It follows from Corollary 3.6 applied to ${}_RP$ and P^*_R.

REMARK. The result of [20] that if R is PF and P is a finitely generated projective RZ-module, then $\mathrm{End}({}_RP)$ is PF, follows also from Corollary 3.9 using the fact that in this case P^* is also a RZ-module by [20, Prop. 1.3].

A ring R is said to be a left FPF ring if every finitely generated faithful left R-module is a generator [3]. In [15] a finitely generated projective module P is called a FPF module if every finitely P-generated module that cogenerates P generates P and P is T-faithful. Then, in [15, Theorem 4] it is shown that if P is a finitely generated projective T-faithful module, then P FPF implies that $S = \mathrm{End}({}_RP)$ is left FPF, and furthermore, if P is a self-generator, then the converse holds. The following result is more general and shows that the hypothesis that P is a self-generator is not needed in Page's result.

THEOREM 3.10. *Let M be a Σ-quasiprojective module. If S is a left* FPF *ring, then every finitely M-generated module that cogenerates $\bar{M}$ generates $\bar{M}$. If M is finitely generated, then the converse holds.*

PROOF. Consider the Gabriel topology $\mathscr{F} = \{I \subset {}_SS \mid MI = M\}$. By Theorem 1.3 there is an equivalence $\mathrm{Hom}_R(M, -) : \mathrm{GF}[M] \to (S, \mathscr{F})$-Mod. Since M is Σ-quasiprojective, it is a projective object of $\sigma[M]$ and hence $\bar{M}$ is a projective object of $\mathrm{GF}[M]$; therefore, S is a projective object of $(S, \mathscr{F})$-Mod and by [6, Coroll. 2.3] the inclusion functor of $(S, \mathscr{F})$-Mod in S-Mod is exact. By using this fact, we see that if S is left FPF then the category $(S, \mathscr{F})$-Mod has the property that each object which is a quotient of some S^n and cogenerates S generates S. This property may be transferred to $\mathrm{GF}[M]$ through the above equivalence of categories and thus we have that each finitely $\bar{M}$-generated module of $\mathrm{GF}[M]$ which cogenerates $\bar{M}$ generates $\bar{M}$. If X is a finitely M-generated module which cogenerates $\bar{M}$, then $\bar{X}$ is finitely $\bar{M}$-generated and, as in the proof of Theorem 3.5, (i)$\Leftrightarrow$(v), we see that $\bar{X}$ cogenerates $\bar{M}$. Therefore, $\bar{X}$ generates $\bar{M}$, and so X generates $\bar{M}$.

For the converse observe that if M is finitely generated, then $\mathrm{Hom}_R(M, -)$: GF[M] $\to$ S-Mod is an equivalence of categories, in which S corresponds to $\bar{M}$. Our hypothesis clearly implies that any object of GF[M] which is finitely $\bar{M}$-generated and cogenerates $\bar{M}$ in GF[M] generates $\bar{M}$ in GF[M]. By the above equivalence we have that S is left FPF.

REMARK. The endomorphism ring of a nonfinitely generated M-faithful Σ-quasiprojective module M such that each finitely M-generated module which cogenerates M generates M need not be left FPF. For instance, the endomorphism ring of an infinite dimensional vector space over a field is never left FPF [3, p. 3.13].

We recall that a ring R is said to be left TCE [13] if the class of torsionless left R-modules is closed under extensions. In [13, Théorème 1.1], it is shown that if P is a projective generator of R-Mod, then R is left TCE if and only if so is $S = \mathrm{End}(_RP)$. We are going to give a more general result.

PROPOSITION 3.11. *Let M be a Σ-quasiprojective module and $S =$ $\mathrm{End}(_RM)$. The the following conditions are equivalent*:

(i) *S is left* TCE.

(ii) *If $0 \to X \to Y \to Z \to 0$ is an exact sequence in R*-Mod *such that Y is M-generated and M-faithful, and X_M and Z are cogenerated by $\bar{M}$, then Y is cogenerated by $\bar{M}$.*

PROOF. Let $\mathscr{F} = \{I \subset {}_SS \mid MI = M\}$. Then, in a similar way to the proof of [13, Théorème 1.1] it can be shown that S is left TCE if and only if the class of $\mathscr{F}$-closed torsionless S-modules is closed under extensions in $(S, \mathscr{F})$-Mod. Since being torsionless for a $\mathscr{F}$-closed module is exactly the same as being cogenerated by S in $(S, \mathscr{F})$-Mod, we may use the equivalence of categories between GF[M] and $(S, \mathscr{F})$-Mod to transfer this property to GF[M]. We get thus that S is left TCE if and only if the class of objects of GF[M] which are cogenerated by $\bar{M}$ is closed under extensions (in GF[M]). But a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ with X, Y, Z in GF[M] is short exact in GF[M] if and only if f is injective, g is surjective, $g \circ f = 0$ and $\mathrm{Ker}\, g/\mathrm{Im}\, f$ is torsion in $\sigma[M]$, that is, if and only if there is a short exact sequence in R-Mod $0 \to K \xrightarrow{f'} Y \xrightarrow{g} Z \to 0$ such that $X = K_M$ and $f'|_X = f$. Since the product of copies of $\bar{M}$ in GF[M] is given by $\Pi_I\bar{M} = (\bar{M}^I)_M$ (where $\bar{M}^I$ is the product in R-Mod and the projections are the obvious ones), we see that the class of objects of GF[M] that are cogenerated by $\bar{M}$ in GF[M] is closed under extensions of GF[M] if and only if condition (ii) holds, and so the proof is complete.

COROLLARY 3.12. *Let M be a Σ-quasiprojective self-generator. Then S is left* TCE *if and only if the class of modules cogenerated by M is closed under extensions in $\sigma[M]$.*

Recall that a ring R is called left SZD if, for every pair of left R-modules N, M, such that $N \subset M$, $\mathrm{Hom}_R(M, R) = 0$ implies $\mathrm{Hom}_R(N, R) = 0$ (see [13]). The following result extends [13, Théorème 1.2].

PROPOSITION 3.13. *Let M be a Σ-quasiprojective module. Then S is left* SZD *if and only if for M-generated M-faithful modules X, Y, such that $X \subset Y$,* $\mathrm{Hom}_R(Y, \bar{M}) = 0$ *implies* $\mathrm{Hom}_R(X, \bar{M}) = 0$.

PROOF. Let $\mathscr{F} = \{I \subset {}_S S \mid MI = M\}$ and $(S, \mathscr{F})$-Mod the quotient category corresponding to $\mathscr{F}$. As in the proof of [13, Théorème 1.2] we see that S is left SZD if and only if for $\mathscr{F}$-closed modules K, L, with $K \subset L$ $\mathrm{Hom}_S(L, S) = 0$ implies $\mathrm{Hom}_S(K, S) = 0$. By the equivalence of categories between $(S, \mathscr{F})$-Mod and GF[M], we see that S is left SZD if and only if GF[M] verifies the stated property.

A ring R is called left QF-3′ if $E({}_R R)$ is a torsionless R-module. As it was remarked in [11] a ring R is left QF-3′ if and only if it is left TCE and left SZD. Thus Propositions 3.11 and 3.13 give a characterization of Σ-quasiprojective modules which have left QF-3′ endomorphism ring. A different characterization is the following.

THEOREM 3.14. *Let M be a Σ-quasiprojective module. Then the following conditions are equivalent:*

(i) *S is left* QF-3′.

(ii) *$E(\bar{M})_M$ is cogenerated by $\bar{M}$.*

(iii) *The class of M-generated modules which are cogenerated by $\bar{M}$ is closed under essential extensions.*

PROOF. (i)$\Leftrightarrow$(ii) Let $\mathscr{F} = \{I \subset {}_S S \mid MI = M\}$. Since S is $\mathscr{F}$-closed, its injective envelope is the same in $(S, \mathscr{F})$-Mod as in S-Mod [19, Prop. 1.7, p. 215] and since the inclusion functor from $(S, \mathscr{F})$-Mod to S-Mod preserves products (as it is left adjoint of the functor **a**) we see that S is left QF-3′ if and only if $E(S)$ is cogenerated by S in $(S, \mathscr{F})$-Mod. By the equivalence between

GF[M] and $(S, \mathcal{F})$-Mod in which $\bar{M}$ corresponds to S, we have that (i) is equivalent to the injective envelope of $\bar{M}$ in GF[M] being cogenerated by $\bar{M}$ in this category. The injective envelope of $\bar{M}$ in GF[M] is $E(\bar{M})_M$ and since the product in GF[M] is the largest M-generated submodule of the product in R-Mod, it is also clear that $E(\bar{M})_M$ is cogenerated by $\bar{M}$ in GF[M] if and only if it is cogenerated by $\bar{M}$ in R-Mod.

(i)$\Leftrightarrow$(iii) By [11, Prop. 1], S is left QF-3′ if and only if the class of torsionless left S-modules is closed under essential extensions. We claim that this property is equivalent to the fact that the class of torsionless $\mathcal{F}$-closed left S-modules is closed under essential extensions in $(S, \mathcal{F})$-Mod (where $\mathcal{F} = \{I \subset {}_S S \mid MI = M\}$). Assume that S is left QF-3′ and let $X \rightarrow Y$ be an essential monomorphism in $(S, \mathcal{F})$-Mod with X torsionless. If Z is a nonzero submodule of Y, then $Z^c = \{x \in Y \mid (Z : x) \in \mathcal{F}\}$ is a nonzero $\mathcal{F}$-closed submodule of Y [7, 5.4] and thus $X \cap Z^c \neq 0$. But $X \cap Z^c = X^c \cap Z^c = (X \cap Z)^c$ and so we have $X \cap Z \neq 0$, so that Y is also an essential extension of X in S-Mod. Therefore Y is torsionless. Conversely, if essential extensions in $(S, \mathcal{F})$-Mod of torsionless $\mathcal{F}$-closed modules are torsionless and L is an essential extension of a torsionless left S-module K, then applying the functor $\mathbf{a} : S\text{-Mod} \rightarrow (S, \mathcal{F})\text{-Mod}$ to a monomorphism from K to S^J and using the facts that $\mathbf{a}$ is left exact and $\mathbf{a}(S^J) \simeq S^J$, we get a monomorphism (in $(S, \mathcal{F})$-Mod and in S-Mod) from $\mathbf{a}(K)$ to S^J. Also, since K is $\mathcal{F}$-torsionfree (because S is $\mathcal{F}$-torsionfree) and essential in L, it is a well known fact that $\mathbf{a}(K)$ is essential (as a subobject in $(S, \mathcal{F})$-Mod and also as a S-submodule) in $\mathbf{a}(L)$. Thus our hypothesis implies that $\mathbf{a}(L)$ is torsionless and since L, being $\mathcal{F}$-torsionfree, is a submodule of $\mathbf{a}(L)$ we get that L is also torsionless.

Now, using the equivalence between GF[M] and $(S, \mathcal{F})$-Mod we see that GF[M] has the property that the class of objects cogenerated by $\bar{M}$ is closed under essential extensions in GF[M]. Since each nonzero submodule of a M-faithful module contains a nonzero M-generated submodule, it is clear that this condition is equivalent to (iii) and the proof is complete.

COROLLARY 3.15. *Let M and N be Σ-quasiprojective modules which generate each other. Then* End$({}_RM)$ *is left* QF-3′ *if and only if* End$({}_RN)$ *is left* QF-3′.

PROOF. We have that in this case $\sigma[M] = \sigma[N]$ and the torsion theories defined by M and N in $\sigma[M]$ are clearly the same. Moreover, $\bar{M}$ and $\bar{N}$ are both projective generators of GF[M] and hence they generate each other in GF[M]. Therefore, condition (iii) of Theorem 3.14 holds for M if and only if it holds for N and thus the result follows.

REMARK. If in particular we take in Corollary 3.15 $N = {}_RR$ and M a projective generator of R-Mod, then we get Corollaire 1.3 of [13]. As a consequence of Propositions 3.11 and 3.13 we have that a similar result to Corollary 3.15 holds for left TCE rings and left SZD rings.

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